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Force-free fields from Hertz potentials

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Abstract. We show a correspondence between magnetic fields that are eigenvectors of the curl operator and certain solutions to the source-free Maxwell equations. When the eigenvalue of the curl is non-constant then the corresponding Maxwell system must be solved in a non-flat spacetime. We show how the generalized Hertz potential method of solving Maxwell's equations in curved spacetimes can be applied to determine eigenvectors of the curl. In the case of a constant eigenvalue the Chandrasekhar–Kendall eigenfunctions are recovered. For non-constant eigenvalue we use the formalism to produce some new force-free fields.

1. Introduction

A force-free field $\boldsymbol{B}$ is a three-dimensional divergenceless eigenfunction of the curl operator

$$\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B} \qquad \text{and} \qquad \nabla \cdot \boldsymbol{B} = 0 \tag{1}$$

where in general the eigenvalue λ may be a function of position. For constant eigenvalues, the eigenfunctions of the curl are necessarily divergenceless and in this case the problem is readily solved [1], although there is still considerable interest in the details of this solution [2–8]. For non-constant eigenvalues the problem is much harder and only isolated solutions, mostly due to Low [9–11], are known. The curl equation no longer guarantees that $\boldsymbol{B}$ is divergenceless, and so this must be imposed as a separate condition. If $\boldsymbol{B}$ is an eigenfunction of the curl operator, then $\nabla \cdot \boldsymbol{B} = 0$ is equivalent to $\nabla\lambda \cdot \boldsymbol{B} = 0$, and so force-free fields with non-constant λ are constrained to lie in the level surfaces of λ.

Discussions of force-free fields are found mostly in the astrophysical literature, where they have been used to model the magnetic field in the solar corona, but the force-free field equations appear in several other branches of physics. For example, the same equations describe the flow of an incompressible fluid for which the vorticity is proportional to the velocity—a so called Beltrami flow. For examples of these applications see the references in [4, 6].

In 1957, Chandrasekhar and Kendall [1] showed how solutions of the constant eigenvalue force-free field equation could be written in terms of solutions of the scalar Helmholtz equation and a specially chosen vector field. This construction is reminiscent of Bromwich, Debye and Whittaker's ansatz for the Hertz potential (details can be found in Nisbet [12]), which allows solutions to Maxwell's equations in a vacuum to be written in terms of these same special vector fields and two solutions to the scalar wave equation. If the time dependence of the Maxwell field F is given by $e^{i\omega t}$, the scalar wave equation

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reduces to the Helmholtz equation and the electric and magnetic fields making up the self-dual and anti-self-dual parts of F, as given by a suitable Debye potential scheme, have exactly the form of the Chandrasekhar–Kendall force-free field solution with eigenvalues $-\omega$ and $+\omega$, respectively.

The purpose of this paper is to show that solving $\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B}$, with λ not necessarily constant, is formally equivalent to solving a source-free Maxwell problem in a background spacetime that has conformally flat instantaneous space-like slices. This allows the machinery that has been developed for solving Maxwell's source-free equations in a non-flat background spacetime, and in particular the curved space generalization of the Debye potential, to be applied to this problem. In section 4 we first apply the Debye potential method to reproduce Chandrasekhar and Kendall's solution, and then use the formalism developed to produce some new non-constant eigenvalue force-free fields.

2. Eigenfunctions of the curl operator as solutions to Maxwell's source-free equations

A source-free electromagnetic field on a four-dimensional Lorentzian manifold is a 2-form F that is both closed and co-closed

$$DF = 0 \qquad \text{and} \qquad D^*F = 0. \tag{2}$$

Here we have used D for the exterior derivative on differential forms and $*$ for the Hodge duality operator associated with the metric tensor G. The operator D^* is the exterior co-derivative and is given in terms of D and $*$, when acting on p-forms, by $\varepsilon(-1)^{(p+1)n+1} * D\, *$, where ε is $+1$ for positive definite metrics and -1 for Lorentzian metrics.

The fact that on a four-dimensional Lorentzian manifold $*$ squares to -1 and maps 2-forms to 2-forms allows F to be split into its self-dual and anti-self-dual parts, $F = F^+ + F^-$, where $2F^\pm = F \mp \mathrm{i} * F$, which have the property $* F^\pm = \pm \mathrm{i} F^\pm$. Both the self-dual and anti-self-dual parts separately satisfy Maxwell's equations, which for these fields can be written simply as $DF^\pm = 0$.

We consider a product spacetime with the product metric $G = -dt^2 + g$ (g is independent of t), and look for $F^\pm$ satisfying $\dot{F}^\pm = \mathrm{i}\omega F^\pm$. Here we use a dot over a symbol to denote the covariant derivative with respect to ∂_t. If F^+ and F^- are decomposed into their associated three-dimensional electric and magnetic vectors, we will find that on the instantaneous space-like slices these vectors satisfy a curved space generalization of the force-free field equations (1) with constant eigenvalues $-\omega$ and $+\omega$ respectively. Then, when g is conformally flat, and the fields $F^\pm$ are suitably restricted, non-constant eigenvalue force-free fields can be constructed.

By the 3-vectors $\boldsymbol{E}$ and $\boldsymbol{B}$ we mean 4-vectors with no ∂_t component, i.e. $dt(\boldsymbol{E}) = dt(\boldsymbol{B}) = 0$. Then g is a metric on 3-vectors and it has an associated Hodge dual $\star$ that acts on p-forms that have no dt components. The relationship between $*$ and $\star$ determines how the orientation on the whole spacetime is related to that on the space-like slices, and we will take

$$* 1 = dt \wedge \star 1.$$

In order to extract a three-dimensional equation, we define an exterior derivative d by

$$D\alpha = dt \wedge \dot{\alpha} + d\alpha.$$

Since the Maxwell fields $F^\pm$ have electric and magnetic 3-vectors related by $\boldsymbol{E} = \pm \mathrm{i} \boldsymbol{B}$, we can dispense with $\boldsymbol{E}$ and write $F^\pm$ in terms B, the dual of $\boldsymbol{B}$, a magnetic field with harmonic time dependence ($\dot{B} = \mathrm{i}\omega B$). If we write

$$F^\pm = \pm \mathrm{i} B \wedge dt + \star B$$

then Maxwell's equations, $DF^{\pm} = 0$, reduce to

$$\star\, dB = \mp\omega B. \tag{3}$$

If g is conformally related to the flat metric $\overline{g}$ by $g = \mathrm{e}^{2\sigma}\overline{g}$, the Hodge star $\bar{\star}$ associated with $\overline{g}$, when acting on 2-forms (like dB) is given by $\bar{\star} = \mathrm{e}^{\sigma}\,\star$. Now we can write equation (3) as

$$\bar{\star}\, dB = \mp\omega \mathrm{e}^{\sigma} B. \tag{4}$$

This can be translated into the notation of the 3-vector calculus by noting that $\bar{\star}\, dB$ is dual to $\boldsymbol{\nabla}\times\boldsymbol{B}$, and so $\boldsymbol{B}$ is an eigenfunction of the curl with eigenvalue $\mp\omega\mathrm{e}^{\sigma}$,

$$\boldsymbol{\nabla}\times\boldsymbol{B} = \mp\omega\mathrm{e}^{\sigma}\boldsymbol{B}.$$

Since the above procedure can be reversed, any eigenfunction of the curl provides a solution to Maxwell's source-free equations in a suitably chosen background spacetime. Hence for a particular eigenvalue, finding eigenfunctions of the curl operator is equivalent to solving a Maxwell problem for a self-dual or anti-self-dual field of a single frequency. A consequence of Maxwell's equations (2) is that B is divergenceless with respect to the conformally flat metric g, $d^{\star}B = 0$, but this is not the same as $d^{\bar{\star}}B = 0$, or equivalently $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$ (unless $\sigma =$ constant), and so we must impose this condition separately if we wish to find force-free fields.

3. Eigenfunctions of the curl from a Debye potential

In the previous section we demonstrated the equivalence of certain solutions of Maxwell's source-free equations to eigenfunctions of the curl, and how this equivalence holds even for non-constant eigenvalues if we consider the appropriate non-flat background spacetime. However, the problem of solving the equations still remains. The form of the Chandrasekhar–Kendall eigenfunctions, that solves this problem for a constant eigenvalue, reduces the problem to a single scalar equation. In recent years, significant progress has been made in the understanding of such methods, and in this section we apply a formalism that achieves such a reduction when the background spacetime admits a privileged tensor; a conformal Killing–Yano 2-form.

In what follows, we will denote the Laplace–Beltrami operator, curvature scalar, conformal 2-forms and Ricci 1-forms associated with G by $\triangle$, $\mathcal{R}$, C_{ab} and P_a, and use the conventions of Benn and Tucker [13] for these and other symbols. We will choose the basis of vector fields to which the labels on C_{ab} and P_a refer to be $\{\partial_t, X_i\}$ so that $dt(X_i) = 0$ and $i = 1, 2, 3$. With this choice, the Ricci 1-forms P_i for G are the same as those for g, with P_0 identically zero (where $X_0 = \partial_t$). As a result, the curvature scalars associated with g and G are also equal and so we need not specify whether we mean three-dimensional or four-dimensional quantities when we use P_i and $\mathcal{R}$. We will denote the 1-form G-metric dual to the vector field X by $X^{\flat}$, and the vector G-metric dual to the 1-form α by $\alpha^{\sharp}$. Since G and g, when acting on vectors with no ∂_t component, are the same, we can also use the symbols $^{\flat}$ and $^{\sharp}$ to denote the g-metric duals of 3-vectors and three-dimensional 1-forms.

3.1. The Debye potential formalism

To construct an anti-self-dual Maxwell field, we will use a self-dual 2-form P^{+} satisfying the conformal Killing–Yano 2-form equation [14, 15]

$$3\nabla_X P^{+} = X \lrcorner\, DP^{+} - X^{\flat}\wedge D^{*}P^{+} \qquad \forall\ X. \tag{5}$$

As a consequence of equation (5), P^+ is necessarily an eigenform of the conformal curvature

$$C \cdot P^+ \equiv \tfrac{1}{2} X^a \lrcorner X^b \lrcorner P^+ C_{ba} = \mu P^+ \tag{6}$$

for some eigenvalue μ, and satisfies the additional integrability condition

$$C \cdot P^+ = \tfrac{1}{3}\left[\triangle P^+ + \tfrac{1}{2}\mathcal{R}P^+\right]. \tag{7}$$

Using equations (5), (6) and (7), it can be shown [15] that given a scalar Debye potential f satisfying

$$\triangle f - \tfrac{1}{6}\mathcal{R}f = -\mu f \tag{8}$$

an anti-self-dual Maxwell field F^- is given by

$$F^- = \mathrm{i}D\left[\tfrac{2}{3} f D^* P - D^*(fP)\right].$$

To solve equations (5) and (8) on a product background, it is convenient to rewrite them in terms of differential forms and operators on the three-dimensional instantaneous slices. First we write P^+ in terms of the three-dimensional 1-form α ($dt(\alpha) = 0$) as

$$P = dt \wedge \alpha + \mathrm{i} \star \alpha.$$

We can impose the harmonic time-dependence on the solution by requiring that there are some constants ω_f and ω_α, such that

$$\dot{f} = \mathrm{i}\omega_f f \qquad \text{and} \qquad \dot{\alpha} = \mathrm{i}\omega_\alpha \alpha.$$

By considering $X = U$, where $dt(U) = 0$, we find that α must satisfy the three-dimensional equation

$$\nabla_U \alpha = \tfrac{1}{2} U \lrcorner d\alpha - \tfrac{1}{3} d^\star \alpha U^\flat \quad \forall\ U \qquad \text{such that} \ \ dt(U) = 0. \tag{9}$$

That is, α is a 'time dependent' 1-form which, for all values of t, is a conformal Killing 1-form with respect to g. Letting $X = \partial_t$ we find that α must also satisfy the additional condition $2\dot{\alpha} = \mathrm{i} \star d\alpha$ which, given the assumed harmonic time-dependence of α, becomes

$$2\omega_\alpha \alpha = \star\, d\alpha. \tag{10}$$

Since P^+ is an eigenform of the conformal curvature, it follows that α is an eigenform of the Ricci curvature

$$P \cdot \alpha \equiv \alpha(X^i) P_i = \nu\alpha \tag{11}$$

where $\nu = \frac{1}{3}\mathcal{R} - 2\mu$. The integrability condition (7) for P^+ is equivalent to the integrability condition for α

$$P \cdot \alpha = \tfrac{1}{2} d^\star d\alpha + \tfrac{2}{3} d d^\star \alpha. \tag{12}$$

To write equation (8) in three-dimensional form, we first denote the Laplace–Beltrami operator on the space-like slices as $\vartriangle$. Then decomposing $\triangle$ as $\vartriangle - \partial_t{}^2$ gives

$$\vartriangle f - \tfrac{1}{8}\mathcal{R} f = \left(\tfrac{1}{24}\mathcal{R} - \omega_f{}^2 - \mu\right) f. \tag{13}$$

We have chosen to split the term involving $\mathcal{R}f$ so that the left-hand side of equation (13) is the conformally-covariant Laplace–Beltrami operator in three dimensions.

If the equations for f and α can be solved, then we can obtain solutions to (4) with $\omega = \omega_f + \omega_\alpha$ by extracting B from F^-, giving

$$\begin{aligned} B &= \star\left[\partial_t \lrcorner (dt \wedge F^-)\right] \\ &= (\omega_f - \omega_\alpha) \star d(f\alpha) + \star\, d \star d(f\alpha). \end{aligned} \tag{14}$$

In terms of the Hodge star of the flat metric $\bar{g}$

$$B = (\omega_f - \omega_\alpha)\mathrm{e}^{-\sigma}\,\bar{\star}\, d(f\alpha) + \mathrm{e}^{-\sigma}\,\bar{\star}\, d(\mathrm{e}^{-\sigma}\,\bar{\star}\, d(f\alpha))$$

and so in the notation of the 3-vector calculus we have

$$\boldsymbol{B} = (\omega_f - \omega_\alpha)\mathrm{e}^{-\sigma}\boldsymbol{\nabla} \times (f\boldsymbol{\alpha}) + \mathrm{e}^{-\sigma}\boldsymbol{\nabla} \times (\mathrm{e}^{-\sigma}\boldsymbol{\nabla} \times (f\boldsymbol{\alpha})). \tag{15}$$

3.2. Solving the three-dimensional equations

In finding suitable 1-forms α and scalars f, the conformal covariance of the three-dimensional equations (9) and (13) for α and f are of considerable utility. Firstly, equation (13) can be solved by $f = \mathrm{e}^{-\frac{1}{2}\sigma}\Psi$, where Ψ satisfies

$$\nabla^2\Psi = \left(\tfrac{1}{24}\mathcal{R} - {\omega_f}^2 - \mu\right)\mathrm{e}^{2\sigma}\Psi \tag{16}$$

and ∇^2 is the three-dimensional flat space Laplacian. Since the conformal Killing 1-forms α are related by the metric g to conformal Killing vectors, solutions to equation (9) are simply scalings of the flat-space conformal Killing 1-forms $\bar{\alpha}$:

$$\alpha = \mathrm{e}^{2\sigma}\bar{\alpha}.$$

However, in general most of these solutions are ruled out by (10), which is not conformally covariant and must be considered separately. Except in a few simple cases, considerable work can be saved by considering the consequences of the integrability conditions (11) and (12). First we note that (10) implies that α is either closed or co-closed for $\omega_\alpha = 0$ and $\omega_\alpha \neq 0$, respectively. Combining equations (10), (11) and (12) gives

$$2{\omega_\alpha}^2\alpha + \tfrac{2}{3}dd^\star\alpha = \nu\alpha \tag{17}$$

which when $\omega_\alpha \neq 0$ reads $2{\omega_\alpha}^2\alpha = \nu\alpha$ and so the eigenvalue of the Ricci curvature must be constant. For $\omega_\alpha = 0$, taking the exterior derivative of equation (17) gives $d\nu \wedge \alpha = 0$, and so the gradient of the Ricci eigenvalue must be proportional to the corresponding eigenvector.

Although equation (16) is a flat space equation, the quantities $\mathcal{R}$ and μ are associated with the conformally-flat metric g. The curvature scalar $\mathcal{R}$ can be written in terms of the conformal factor and the flat gradient and Laplacian as

$$\mathcal{R} = -\mathrm{e}^{-2\sigma}\left(2|\boldsymbol{\nabla}\sigma|^2 + 4\nabla^2\sigma\right) \tag{18}$$

and using equations (11) and (12) the eigenvalue μ can be written as

$$\begin{aligned}\mu &= \frac{1}{6}\mathcal{R} - \frac{1}{2\bar{g}(\alpha^\sharp,\alpha^\sharp)}\;\bar{g}\left(\left[\frac{1}{2}d^\star d\alpha + \frac{2}{3}dd^\star\alpha\right]^\sharp, \alpha^\sharp\right)\\ &= \frac{1}{6}\mathcal{R} - \frac{1}{2|\boldsymbol{\alpha}|^2}\left[\frac{1}{2}\mathrm{e}^{-\sigma}\boldsymbol{\nabla}\times\left(\mathrm{e}^{-\sigma}\left(\boldsymbol{\nabla}\times\boldsymbol{\alpha}\right)\right) - \frac{2}{3}\boldsymbol{\nabla}\left(\mathrm{e}^{-3\sigma}\boldsymbol{\nabla}\cdot(\mathrm{e}^{\sigma}\boldsymbol{\alpha})\right)\right]\cdot\boldsymbol{\alpha}.\end{aligned} \tag{19}$$

4. Solutions to the force-free field equations

As the first step in using the method described in the previous section we need to find an α that satisfies both the conformal Killing equation (9) and the additional constraint (10). The general solution to the conformal Killing equation can easily be written down as $\mathrm{e}^{2\sigma}$ times a linear combination of 10 independent flat-space conformal Killing 1-forms. For some choices of σ, $\star d$ of a conformal Killing 1-form is also a conformal Killing 1-form (possibly zero) and then it is a simple matter to determine the general solution to both (9) and (10). In general it is not that easy, but nevertheless some solutions can be constructed directly from flat-space conformal Killing 1-forms. If we take $\overline{\alpha}$ to be a conformal Killing 1-form for $\overline{g}$ of the form $s\,du$, where s and u are some scalar functions, and then consider $g = s^{-1}w\overline{g}$, where w is an arbitrary function of u, we find that $\alpha = w\,du$ is a conformal Killing 1-form for g which clearly satisfies (10) for $\omega_\alpha = 0$. In this section, we present some solutions that can be found in this way.

4.1. Chandrasekhar–Kendall eigenfunctions

To solve the force-free field equations for constant eigenvalues, we set $\sigma = 0$ and so g is flat, for which the solutions to the conformal Killing equation (9) are well known. In flat space, $\star d$ of a conformal Killing 1-form is also conformal Killing and so it is then a simple matter to see that the only solutions to (10) are the the closed conformal Killing 1-forms, that is, the generators of translations and dilations. Since $d\alpha = 0$ we have $\omega_\alpha = 0$ and so we can take $\omega = \omega_f$. Equation (13) becomes the scalar Helmholtz equation

$$\nabla^2 f + \omega^2 f = 0$$

and equation (15) gives Chandrasekhar and Kendall's solution [1]

$$\boldsymbol{B} = \omega \boldsymbol{\nabla} \times (f\boldsymbol{\alpha}) + \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (f\boldsymbol{\alpha})$$

where $\boldsymbol{\alpha}$ is any constant vector or the radius vector $\boldsymbol{r}$. Since for constant-eigenvalue eigenfunctions the problem is solved in a flat spacetime, the Chandrasekhar–Kendall eigenfunctions could have been obtained directly using the 3-vector methods of Debye and Whittaker.

4.2. Eigenvalue depending on one Cartesian coordinate x

Given an eigenvalue that depends only on one Cartesian coordinate x, we can construct a class of force-free fields from the harmonic functions that depend only on the other two coordinates in the following manner.

Equations (9) and (10) can be solved immediately by $\alpha = \mathrm{e}^{2\sigma} dx$ and again we have $\omega = \omega_f$. After calculating $\mathcal{R}$ and μ using (18) and (19), equation (16) becomes

$$\nabla^2 \Psi = \left(\tfrac{1}{4}\sigma'^2 - \tfrac{1}{2}\sigma'' - \omega^2 \mathrm{e}^{2\sigma}\right)\Psi. \tag{20}$$

The divergence-free property of B is equivalent to B having no x-component, and by writing (15) out explicitly, we see that this requires that Ψ satisfy

$$\Psi_{yy} + \Psi_{zz} = 0.$$

We can solve for Ψ by separation of variables, giving the solution

$$\Psi = \Phi \mathrm{e}^{-\frac{1}{2}\sigma}\left[A \sin\left(\omega \int \mathrm{e}^{\sigma}\, dx\right) + B \cos\left(\omega \int \mathrm{e}^{\sigma}\, dx\right)\right]$$

where Φ is a harmonic function of y and z, and A and B are arbitrary constants. So in 3-vector form, after dividing through by ω, we find that divergenceless solutions to $\boldsymbol{\nabla} \times \boldsymbol{B} = \lambda(x)\boldsymbol{B}$ may be written as

$$\begin{aligned}\boldsymbol{B} &= p\boldsymbol{\nabla} \times (\Phi\hat{\boldsymbol{x}}) + \lambda^{-1}\boldsymbol{\nabla} \times \left(p\boldsymbol{\nabla} \times (\Phi\hat{\boldsymbol{x}})\right)\\ &= \left(0, \Phi_z p + \Phi_y p', -\Phi_y p + \Phi_z p'\right)\end{aligned}$$

where $p = A \sin \int \lambda\, dx + B \cos \int \lambda\, dx$. In general, linear combinations of solutions are also solutions.

4.3. Eigenvalue depending on the spherical radial coordinate r

A similar result to the Cartesian case can be obtained for eigenvalues depending only on the spherical radial coordinate r. As in the previous case, (9) and (10) can be solved by inspection with $\alpha = e^{2\sigma} r\,dr$ and $\omega = \omega_f$. Ψ again satisfies equation (20), except that now σ is a function of r not x. Again, we proceed by separation of variables, with $\Psi = R(r)\Phi(\theta, \phi)$. The equation $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$ leads to

$$\nabla^2 \Phi = 0$$

for which the only solution, regular on the whole sphere, is $\Phi =$ constant. However, if one ray from the origin is excluded, solutions can be found and are given by

$$\Phi_n = (A_n \sin n\phi + B_n \cos n\phi) \tan^n(\tfrac{1}{2}\theta) \qquad n \neq 0 \tag{21}$$

$$\Phi_0 = A_0 + B_0 \log \tan(\tfrac{1}{2}\theta). \tag{22}$$

The solution to the radial equation

$$R'' + \frac{2}{r} R' + \left(\frac{1}{4}\sigma'^2 - \frac{1}{2}\sigma'' - \omega^2 e^{2\sigma} \right) R = 0$$

is

$$R = r^{-1} e^{-\frac{1}{2}\sigma} \sin\left(\omega \int e^{\sigma}\, dr \right).$$

So, after cancelling factors of r and e^{σ} and again dividing through by ω, we find that solutions to equation (1) with $\lambda = \lambda(r)$ can be written as

$$\boldsymbol{B} = \boldsymbol{\nabla} \times (\chi \hat{\boldsymbol{r}}) + \lambda^{-1} \boldsymbol{\nabla} \times \boldsymbol{\nabla} \times (\chi \hat{\boldsymbol{r}}) \tag{23}$$

where

$$\chi = \sum_{n=-\infty}^{\infty} \Phi_n \sin \int \lambda\, dr. \tag{24}$$

For $n = 0$, we have explicitly

$$\chi = \big(A_0 + B_0 \log \tan(\tfrac{1}{2}\theta)\big) \sin \int \lambda\, dr.$$

Taking $B_0 = -b$, we obtain the solution given by Low [11],

$$\boldsymbol{B} = \frac{b}{r \sin\theta} \left(\cos \int \lambda\, dr\, \widehat{\boldsymbol{\theta}} + \sin \int \lambda\, dr\, \widehat{\boldsymbol{\phi}} \right).$$

For the restricted case of λ = constant, the solution obtained from equations (23) and (24) has been given by Zaghoul and Barajas [16]. However, this is just a Chandrasekhar–Kendall eigenfunction that had not been written out explicitly, as it is singular.

4.4. Eigenvalue depending on the cylindrical radial coordinate ρ

It is not possible to find all possible solutions to the force-free field equations by the Debye potential method described in this paper. For example, the well know solution of Low [9] with eigenvalue $2a(1 + a^2\rho^2)^{-1}$, given in cylindrical coordinates by

$$\boldsymbol{B} = \frac{a\rho}{1 + a^2\rho^2} \widehat{\boldsymbol{\phi}} + \frac{1}{1 + a^2\rho^2} \hat{\boldsymbol{z}}$$

cannot be reproduced in this manner, as the product spacetime does not admit a conformal Killing–Yano 2-form. This solution corresponds to a vacuum Maxwell field in a spacetime with metric $G = -dt^2 + 4a^2(1+\rho^2)^{-2}\overline{g}$, for which the Ricci tensor has two distinct eigenvalues of $\frac{1}{2}(3 - a^2\rho^2)$, corresponding to the eigenspace spanned by $\{d\rho, d\phi\}$, and $\frac{1}{2}(1 - a^2\rho^2)$ corresponding to the eigen-1-form dz. Neither eigenvalue is constant and so we must consider only 1-forms proportional to their gradient, that is, proportional to $d\rho$. Since we require that this must be proportional to a flat-space conformal Killing 1-form, and clearly this cannot be the case, we can see that we cannot solve (9) and (10) in this case.

To find an eigenvalue of the curl that leads to a spacetime containing a conformal Killing–Yano 2-form, we can use the fact that the generator of rotations around the z-axis in flat space is dual to $\rho^2 d\phi$. So if we choose $\mathrm{e}^{2\sigma} = \rho^{-2}w(\phi)$ an obvious choice for α is $w(\phi)d\phi$. In order to be able to impose $\boldsymbol{\nabla}\cdot\boldsymbol{B} = 0$, we will make the the more restrictive choice of $\mathrm{e}^{2\sigma} = \rho^{-2}$, in which case we can take $\alpha = d\phi$ and $\omega = \omega_f$. Equation (16) becomes

$$\nabla^2\Psi = \left(\tfrac{1}{4} - \omega^2\right)\rho^{-2}\Psi \tag{25}$$

and for $\boldsymbol{B}$ to be divergenceless requires

$$-\omega\Psi_z + \frac{1}{2\rho}\Psi_\phi + \Psi_{\rho\phi} = 0. \tag{26}$$

We cannot in general satisfy both of these conditions, but if Ψ is a function of ρ alone, then equation (26) is identically satisfied and equation (25) becomes

$$\rho^2\Psi'' + \rho\Psi' + \left(\omega^2 - \tfrac{1}{4}\right)\Psi = 0$$

which has the following solutions:

$$\omega^2 > \tfrac{1}{4} \quad \Rightarrow \quad \Psi = A\cos\left(\sqrt{\omega^2 - \tfrac{1}{4}}\log\rho\right) + B\sin\left(\sqrt{\omega^2 - \tfrac{1}{4}}\log\rho\right)$$

$$\omega^2 < \tfrac{1}{4} \quad \Rightarrow \quad \Psi = A\rho^{\sqrt{\frac{1}{4}-\omega^2}} + B\rho^{-\sqrt{\frac{1}{4}-\omega^2}}$$

$$\omega^2 = \tfrac{1}{4} \quad \Rightarrow \quad \Psi = A\log\rho + B\,.$$

Here we have written the $\omega \neq \frac{1}{4}$ solution in two ways so that it is explicitly real. This can be written in a more compact form if we define a constant q by letting $\omega = \sqrt{-q(q+1)}$. Then a solution to $\boldsymbol{\nabla}\times\boldsymbol{B} = \sqrt{-q(q+1)}\rho^{-1}\boldsymbol{B}$ is

$$\boldsymbol{B} = \rho^q\ \widehat{\boldsymbol{\phi}} + \sqrt{\frac{q+1}{-q}}\rho^q\ \hat{\boldsymbol{z}}.$$

Note that for each value of $\omega \neq \frac{1}{4}$, there are two possible choices for q ($q \to -q-1$ leaves ω unchanged), and so a linear combination of these two solutions is also a solution. For the case of $\omega = \frac{1}{4}$, we have

$$\boldsymbol{B} = A'\frac{\log\rho}{\sqrt{\rho}}\left(\pm\hat{\boldsymbol{z}} - \widehat{\boldsymbol{\phi}}\right) + B'\frac{1}{\sqrt{\rho}}\left(\pm\hat{\boldsymbol{z}} + \widehat{\boldsymbol{\phi}}\right).$$

5. Conclusions

By establishing the equivalence of eigenfunctions of the curl to certain source-free Maxwell solutions we have been able to apply the (generalized) Debye potential scheme to the force-free field problem. When the eigenvalue of the curl is constant we recover the Chandrasekhar–Kendall eigenfunctions. The choice of vector $\hat{\boldsymbol{z}}$ or $\boldsymbol{r}$ in the Chandrasekhar–Kendall eigenfunctions corresponds to choosing either the Whittaker or Debye–Bromwich potential in the equivalent source-free Maxwell problem. It is known that for either of these choices of Debye potential the class of solutions obtained is complete (excluding the Coulomb solution) [12]. In either case, all anti-self-dual fields F^- with $\dot{F} = \mathrm{i}\omega F$ are given by the Debye potential. So we can conclude that, for either of the two choices of vector, the Chandrasekhar–Kendall eigenfunctions of the curl are complete.

When the eigenvalue λ of the curl is not constant then we have an equivalent curved-space source-free Maxwell problem. This can be solved by the generalized Hertz potential scheme. In the case of $\lambda = \lambda(r)$ and $\lambda = \lambda(x)$, we have found a class of solutions that can be expressed in a manner very similar to Chandrasekhar and Kendall's original constant-λ solutions. They are given in terms of a scalar function that is harmonic on the magnetic surfaces, respectively $r =$ constant and $x =$ constant. Whereas many other solutions to the eigenvalue problem can be found, they will not necessarily satisfy $\boldsymbol{\nabla} \cdot \boldsymbol{B} = 0$. In particular, if $\mathrm{e}^{\sigma} = 2(r^2 + c)^{-1}$, then the spacetime is of constant curvature c, and there are many solutions to equations (9) and (10).

It is known that conformal Killing–Yano tensors can only exist in certain algebraically-special spacetimes. This fact can be used to restrict the possible eigenvalues of the curl that can be found by the methods of this paper. It is also possible to reduce Maxwell's equations to a scalar equation under slightly less restrictive conditions than the existence of a conformal Killing–Yano tensor. In principle this could be used to find other force-free fields with non-constant eigenvalue. A detailed report of Debye potential methods for solving the curved spacetime Maxwell equations will be given elsewhere [15].

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